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RANDOM VIBRATIONS OF A MULTI-DEGREE-OF-FREEDOM NON-LINEAR SYSTEM USING THE VOLTERRA SERIES†

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1. INTRODUCTION

In the first part of this study [1], the Volterra series was used to approximate the Frequency Response Function (FRF) of a Sd.o.f. Duffing oscillator system under random excitation. The results were rather interesting; it was found that the non-linear system FRF had a greatly augmented pole-zero structure as compared to the underlying linear system. This served to explain the observed peak in the Duffing oscillator spectrum at three times the natural frequency and also indicated that peaks at higher multiples could be expected. It was also shown that the poles for the non-linear system FRF were all in the upper-half of the complex plane and thus that the system would appear to be linear on applying the Hilbert transform test. The object of the current paper is to extend the analysis to Md.o.f. systems in the simplest way possible. The composite FRF for a 2d.o.f. system with cubic non-linearity is computed for a white Gaussian excitation of spectral density P. The calculation is carried out to $O(P^2)$.

The layout of this Letter is as follows. Section 2 describes how the composite FRF for non-linear system is computed in terms of the Volterra series. Section 3 describes the 2d.o.f. system which forms the basis of the analysis. The pole structure FRF is obtained in section 4 and the results of the calculations are verified by numerical simulation in section 5.

2. THE COMPOSITE FRF

The standard form of the Volterra Series is assumed for an input–output process $x(t) \rightarrow y(t)$ [2],

$$y(t) = y_1(t) + y_2(t) + y_3(t) + \cdots + y_n(t) + \cdots, \tag{1}$$

where

$$y_n(t) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \mathrm{d}\tau_i \cdots \mathrm{d}\tau_n h_n(\tau_1, \ldots, \tau_n) x(t-\tau_1) \cdots x(t-\tau_n). \tag{2}$$

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The functions h_n are the Volterra kernels. The *higher-order FRFs* (HFRFs) or *Volterra kernel transforms*, $H_n(\omega_1, \ldots, \omega_n)$, $n = 1, \ldots, \infty$ are simply the Fourier transforms of the kernels

$$H_n(\omega_1, \ldots, \omega_n) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \mathrm{d}\tau_1 \cdots \mathrm{d}\tau_n \, h_n(\tau_1, \ldots, \tau_n) \mathrm{e}^{-i(\omega_1 \tau_1 + \cdots + \omega_n \tau_n)}. \tag{3}$$

The *composite FRF* for random excitation, $\Lambda_r(\omega)$ is defined by analogy with that for a linear system†,

$$\Lambda_r(\omega) = \frac{S_{yx}(\omega)}{S_{xx}(\omega)} \tag{4}$$

and the term *composite* FRF is used because $\Lambda_r(\omega)$, for a non-linear system, will not generally be equal to $H_1(\omega)$ but will receive contributions from all H_n. It is also dependent upon the characteristics of the input. In the specific case of interest here, the form of the input is fixed as Gaussian white noise ($S_{xx}(\omega) = P$). As a consequence, the FRF depends only on the power spectral density of the input, and it is shown in reference [1] that

$$\Lambda_r(\omega) = \sum_{n=1}^{n=\infty} \frac{(2n)!\, P^{n-1}}{n!\, 2^n (2\pi)^{n-1}} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \mathrm{d}\omega_1 \cdots \mathrm{d}\omega_{n-1} H_{2n-1}$$

$$\times (\omega_1, -\omega_1, \ldots, \cdots \omega_{n-1}, -\omega_{n-1}, \omega). \tag{5}$$

Note that $\Lambda_r(\omega)$ tends to the linear FRF as the power spectral density of the excitation tends to zero.

3. THE Md.o.f. SYSTEM

The system investigated here is a simple 2d.o.f. non-linear system with lumped-mass characteristics. The equations of motion are

$$m\ddot{y}_1 + 2c\dot{y}_1 - c\dot{y}_2 + 2ky_1 - k_2 y_2 + k_3 y_1^3 = x(t),$$

$$m\ddot{y}_2 + 2c\dot{y}_2 - c\dot{y}_1 + 2ky_2 - k_1 y_1 = 0. \tag{6}$$

The underlying linear system is symmetrical but the non-linearity breaks the symmetry and therefore shows itself in both modes (this will be elaborated later). If the FRFs for the processes $x(t) \rightarrow y_1(t)$ and $x(t) \rightarrow y_2(t)$ are denoted $H_1^{(1)}(\omega)$ and $H_1^{(2)}(\omega)$, then it is a straightforward matter to establish that

$$H_1^{(1)}(\omega) = R_1(\omega) + R_2(\omega), \qquad H_1^{(2)}(\omega) = R_1(\omega) - R_2(\omega) \tag{7}$$

†The definition of the FRF of a linear system based on the input/output cross-spectrum, $S_{yx}(\omega)$, and input autospectrum, $S_{xx}(\omega)$, is well known, $H(\omega) = (S_{yx}(\omega)/S_{xx}(\omega))$.

and the R_1 and R_2 are (up to a multiplicative constant) the FRFs of the individual modes

$$R_1(\omega) = \frac{1}{2}\frac{1}{-m(\omega^2 - \omega_{n1}^2) + 2\mathrm{i}\zeta_1\omega_{n1}\omega} = \frac{-1}{2m}\frac{1}{(\omega - p_1)(\omega - p_2)},$$

$$R_2(\omega) = \frac{1}{2}\frac{1}{-m(\omega^2 - \omega_{n2}^2) + 2\mathrm{i}\zeta_2\omega_{n2}\omega} = \frac{-1}{2m}\frac{1}{(\omega - q_1)(\omega - q_2)}, \tag{8}$$

where ω_{n1} and ω_{n2} are the first and second undamped natural frequencies and ζ_1 and ζ_2 are the corresponding dampings. p_1 and p_2 are the poles of the first mode and q_1 and q_2 are the poles of the second mode. According to elementary theory

$$p_1, p_2 = \pm\,\omega_{d1} + \mathrm{i}\zeta_1\omega_{n1}, \qquad q_1, q_2 = \pm\,\omega_{d2} + \mathrm{i}\zeta_2\omega_{n2}, \tag{9}$$

where ω_{d1} and ω_{d2} are the first and second damped natural frequencies.

From this point on, the calculation will concentrate on the FRF $H_1^{(1)}(\omega)$ and the identifying superscript will be omitted, the expressions are always for the process $x(t) \to y_1(t)$.

In order to calculate the FRF up to order $\mathrm{O}(P^2)$ it is necessary to evaluate the kernel transforms $H_1(\omega_1), H_3(\omega_1, \omega_2, \omega_3)$ and $H_5(\omega_1, \omega_2, \omega_3, \omega_4, \omega_5)$. This is accomplished by using the method of harmonic probing [3]. H_1 is given above in the first of equations (8). The calculation for H_3 is a straightforward extension of the Sd.o.f. calculation in reference [1] and the result needed for the composite FRF is

$$H_3(\omega_1, -\omega_1, \omega) = -k_3|H_1(\omega_1)|^2 H_1(\omega)^2. \tag{10}$$

A slightly more lengthy calculation yields

$$\begin{aligned} H_5(\omega_1, -\omega_1, \omega_2, -\omega_2, \omega) = {} & (3k_3^2/10)H_1(\omega)^2|H_1(\omega_1)|^2|H_1(\omega_2)|^2 \\ & \times\{2H_1(\omega) + H_1(\omega_1) + H_1(-\omega_1) + H_1(\omega_2) + H_1(-\omega_2) \\ & + H_1(\omega_1 + \omega_2 + \omega) + H_1(\omega_1 - \omega_2 + \omega) \\ & + H_1(-\omega_1 + \omega_2 + \omega) + H_1(-\omega_1 - \omega_2 + \omega)\}, \end{aligned} \tag{11}$$

and the first three terms in expansion (5) are now within reach:

$$\begin{aligned} \Lambda_r(\omega) = {} & H_1(\omega) + \frac{3P}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}\omega_1\, H_3(\omega_1, -\omega_1, \omega) + \frac{15P^2}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \\ & \times \mathrm{d}\omega_2\,\mathrm{d}\omega_2\, H_5(\omega_1, -\omega_1, \omega_2, -\omega_2, \omega) + \mathrm{O}(P^3). \end{aligned} \tag{12}$$

The simple geometry chosen here results in an identical functional form for $\Lambda_r(\omega)$ in terms of $H_1(\omega)$ as that obtained in reference [1]. The critical difference is now that $H_1(\omega)$ corresponds to a multi-mode system, and this complicates the integrals in equation (5) a little.

4. THE POLE STRUCTURE OF THE COMPOSITE FRF

The first integral which requires evaluation in equation (12) is the order P term,

$$I_1 = -\frac{3k_3 P}{2\pi} H_1(\omega)^2 \int_{-\infty}^{+\infty} \mathrm{d}\omega_1 |H_1(\omega_1)|^2. \tag{13}$$

However, as the integral does not involve the parameter ω, it evaluates to a constant, so the order P term does not introduce any new poles into the FRF but raises the order of the linear system poles.

The order P^2 term requires more effort; this takes the form

$$\frac{S_{y_5x}(\omega)}{S_{xx}(\omega)} = \frac{9P^2 k_3^2 H_1(\omega)^3}{4\pi^2} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{d}\omega_1\, \mathrm{d}\omega_2\, |H_1(\omega_1)|^2 |H_1(\omega_2)|^2 + \frac{9P^2 k_3^2 H_1(\omega)^2}{2\pi^2}$$

$$\times \left\{ \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{d}\omega_1 \mathrm{d}\omega_2\, H_1(\omega_1)|\, H_1(\omega_1)|^2\, |H_1(\omega_2)|^2 \right.$$

$$\left. + \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{d}\omega_1\, \mathrm{d}\omega_2\, H_1(\omega_1 + \omega_2 + \omega)|H_1(\omega_1)|^2\, H_1(\omega_2)|^2 \right\}. \tag{14}$$

The first and second integrals may be dispensed with as they also contain integrals which do not involve ω, and there is no need to give the explicit solution here; no new poles are introduced. The terms simply raise the order of the linear system poles to three again.

The third term in equation (14) is the most complicated. However, it is routinely expressed in terms of 32 integrals I_{jklmn}, where

$$I_{jklmn} = \frac{9P^2 k_3^2\, H_1(\omega)^2}{2\pi^2} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{d}\omega_1\, \mathrm{d}\omega_2\, R_j(\omega_1 + \omega_2 + \omega)$$

$$\times R_k^*(\omega_1) R_l(\omega_1) R_m^*(\omega_2) R_n(\omega_2). \tag{15}$$

In fact, because of the manifest symmetry in ω_1 and ω_2, it follows that

$$I_{jklmn} = I_{jmnkl}, \tag{16}$$

and this reduces the number of independent integrals to 20. A little further thought reveals the relation

$$I_{jklmn} = S[I_{s(j)s(k)s(l)s(m)s(n)}], \tag{17}$$

where the s operator changes the value of the index from 1 to 2 and *vice versa* and the S operator exchanges the subscripts on the constants, i.e. $\omega_{d1} \leftrightarrow \omega_{d2}$, etc. This reduces the number of integrals to 10. It is sufficient to evaluate the following: I_{11111}, I_{11112}, I_{11121}, I_{11122}, I_{11212}, I_{11221}, I_{11222}, I_{12121}, I_{12122} and I_{12222}. Evaluation of the integral is a straightforward exercise in the calculus of residues, which nonetheless requires some help from computer algebra. The expression is rather lengthy and will not be given here; the important point is that the term I_{jklmn} is found to have poles in the positions

$$\pm\,\omega_{dk} \pm \omega_{dl} \pm \omega_{dm} + \mathrm{i}(\omega_{nk}\zeta_k + \omega_{nl}\zeta_l + \omega_{nm}\zeta_m). \tag{18}$$

It transpires, that, as a result of pole-zero cancellation, the number of poles varies for each of the independent integrals. I_{11111} and I_{11112} have simple poles at

$$\omega_{d1} + 3\mathrm{i}\omega_{n1}\zeta_1, \qquad -\,\omega_{d1} + 3\mathrm{i}\omega_{n1}\zeta_1, \qquad 3\omega_{d1} + 3\mathrm{i}\omega_{n1}\zeta_1, \qquad -\,3\omega_{d1} + 3\mathrm{i}\omega_{n1}\zeta_1, \tag{19}$$

so, by the symmetries described above, I_{12222}, amongst others, has poles at

$$\begin{aligned} &\omega_{d2} + 3\mathrm{i}\omega_{n2}\zeta_2, && -\,\omega_{d2} + 3\mathrm{i}\omega_{n2}\zeta_2, \\ &3\omega_{d2} + 3\mathrm{i}\omega_{n2}\zeta_2, && -\,3\omega_{d2} + 3\mathrm{i}\omega_{n2}\zeta_2. \end{aligned} \tag{20}$$

I_{11121}, I_{11122} and I_{11212} have simple poles at

$$\begin{aligned} &\omega_{d2} + \mathrm{i}(2\omega_{n1}\zeta_1 + \omega_{n2}\zeta_2), && -\,\omega_{d2} + \mathrm{i}(2\omega_{n1}\zeta_1 + \omega_{n2}\zeta_2), \\ &2\omega_{d1} + \omega_{d2} + \mathrm{i}(2\omega_{n1}\zeta_1 + \omega_{n2}\zeta_2), && -\,2\omega_{d1} + \omega_{d2} + \mathrm{i}(2\omega_{n1}\zeta_1 + \omega_{n2}\zeta_2), \\ &2\omega_{d1} - \omega_{d2} + \mathrm{i}(2\omega_{n1}\zeta_1 + \omega_{n2}\zeta_2), && -\,2\omega_{d1} - \omega_{d2} + \mathrm{i}(2\omega_{n1}\zeta_1 + \omega_{n2}\zeta_2), \end{aligned} \tag{21}$$

and finally I_{11221}, I_{11222}, I_{12121} and I_{12122} have poles at

$$\begin{aligned} &\omega_{d1} + \mathrm{i}(2\omega_{n2}\zeta_2 + \omega_{n1}\zeta_1), && -\,\omega_{d1} + \mathrm{i}(2\omega_{n2}\zeta_2 + \omega_{n1}\zeta_1), \\ &2\omega_{d2} + \omega_{d1} + \mathrm{i}(2\omega_{n2}\zeta_2 + \omega_{n1}\zeta_1), && -\,2\omega_{d2} + \omega_{d1} + \mathrm{i}(2\omega_{n2}\zeta_2 + \omega_{n1}\zeta_1), \\ &2\omega_{d2} - \omega_{d1} + \mathrm{i}(2\omega_{n2}\zeta_2 + \omega_{n1}\zeta_1), && -\,2\omega_{d2} - \omega_{d1} + \mathrm{i}(2\omega_{n2}\zeta_2 + \omega_{n1}\zeta_1), \end{aligned} \tag{22}$$

and this exhausts all the possibilities.

This calculation motivates the following conjecture. In a Md.o.f. system, the composite FRF from random excitation has poles at all the combination frequencies of the single-mode resonances. This is a pleasing result; there are echoes of the fact that a two-tone periodically excited non-linear Md.o.f. has output components at all the combinations of the input frequencies. A further observation is that all of the poles are in the upper half-plane. This means that the Hilbert transform test will fail to diagnose non-linearity from the FRF [1]. It was observed in reference [1], that in the Sd.o.f. system, each new order in P produced higher multiplicities for the poles leading to the conjecture that the poles are actually isolated essential singularities. It has not been possible to pursue the calculation here to higher orders. The results above do show however, that the multiplicity of the linear system poles appears to be increasing with the order of P in much the same way as for the Sd.o.f. case.

In reference [1], the case of a Duffing oscillator with an additional quadratic non-linearity was considered and it was found that poles occurred at even multiples of the fundamental. It is conjectured on the basis of the results above, that an even non-linearity in a Md.o.f. system will generate poles at all the even sums and differences.

5. VALIDATION

The validation of the results above will be carried out using data from numerical simulation. Consider the linear mass–damper–spring system of Figure 1 which is a simplified version of equation (6). The equations of motion are

$$m\ddot{y}_1 + c\dot{y}_1 + k(2y_1 - y_2) = x_1(t), \qquad m\ddot{y}_2 + c\dot{y}_2 + k(2y_2 - y_1) = x_2(t). \quad (23)$$

The system clearly posesses a certain symmetry. Eigenvalue analysis reveals that the two modes are $(1, 1)^{\mathrm{T}}$ and $(1, -1)^{\mathrm{T}}$. Suppose a cubic non-linearity is added between the two masses, the equation are modified to

$$m\ddot{y}_1 + c\dot{y}_1 + k(2y_1 - y_2) + k_3(y_1 - y_2)^3 = x_1(t),$$

$$m\ddot{y}_2 + c\dot{y}_2 + k(2y_2 - y_1) + k_3(y_2 - y_1)^3 = x_2(t), \quad (24)$$

and the non-linearity couples the two equations. In modal space, the situation is a little different. Changing to normal co-ordinates via

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad (25)$$

yields

$$m\ddot{u}_1 + c\dot{u}_1 + ku_1 = (1/\sqrt{2})(x_1 + x_2) = p_1,$$

$$m\ddot{u}_2 + c\dot{u}_2 + 3ku_2 + 4\sqrt{2}k_3u_2^2 = (1/\sqrt{2})(x_1 - x_2) = p_2. \quad (26)$$

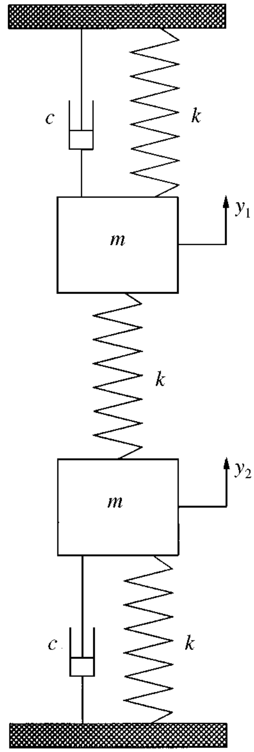

Figure 1. Basic 2d.o.f. linear system.

The system decouples into two Sd.o.f. systems, one linear and one non-linear. This is due to the fact that in the first mode, masses 1 and 2 are moving in phase with constant separation. As a result, the non-linear spring is never exercised and the mode is linear.

Suppose the non-linearity were between the first mass and ground. The equations of motion in physical space would then be

$$m\ddot{y}_1 + c\dot{y}_1 + k(2y_1 - y_2) + k_3 y_1^3 = x_1(t),$$

$$m\ddot{y}_2 + c\dot{y}_2 + k(2y_2 - y_1) = x_2(t), \tag{27}$$

and in modal co-ordinates would be

$$m\ddot{u}_1 + c\dot{u}_1 + ku_1 + (k_3/2)(u_1 + u_2)^3 = p_1,$$

$$m\ddot{u}_2 + c\dot{u}_2 + 3ku_2 - (k_3/2)(u_1 + u_2)^3 = p_2 \tag{28}$$

and the two modes are coupled by the non-linearity.

Both systems (22) and (27) were simulated by using fourth order Runge–Kutta with a slight modification; a quadratic non-linearity was added to the cubic of the

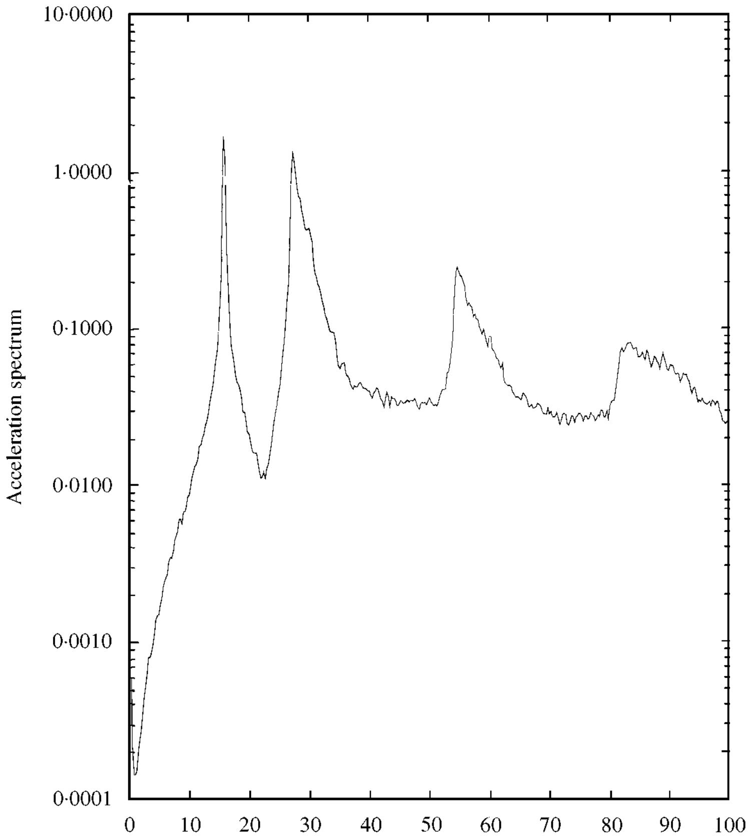

Figure 2. Spectrum from 2d.o.f. system with non-linear spring centred.

form $k_2 y_1^2$ or $k_2(y_1 - y_2)^2$. The values of the parameters were $m = 1, c = 2, k = 10^4$, $k_2 = 10^7$ and $k_3 = 5 \times 10^9$. The excitation x_2 was zero and x_1 initially had r.m.s. 2·0, but this was low-pass filtered into the interval 0 to 100 Hz. With these parameter values the undamped natural frequencies were 15·92 and 27·57 Hz. The sampling frequency was 500 Hz. By using the acceleration response data $\ddot{y}_1$, the output spectra were computed; a 2048-point FFT was used and 100 averages were taken.

Figure 2 shows the output spectrum for system (22). As only the second mode is non-linear, the only additional poles above those for the linear system occur at multiples of the second natural frequency. The presence of the poles is clearly indicated by the peaks in the spectrum at twice and thrice the fundamental.

Figure 3 shows the output spectrum for system (27). Both modes are non-linear and as in the analysis above, poles occur at the sum and differences between the modes. Among the peaks present are: $2\omega_1 \approx 31{\cdot}84$ Hz, $2\omega_2 \approx 55{\cdot}14$, $\omega_2 - \omega_1 \approx 11{\cdot}65, \omega_2 + \omega_1 \approx 55{\cdot}14, 3\omega_1 \approx 47{\cdot}76, 2\omega_1 - \omega_2 \approx 4{\cdot}27$. The approximate nature of the positions is due to the fact that the peaks move as result of the interactions between the poles as discussed in reference [1].

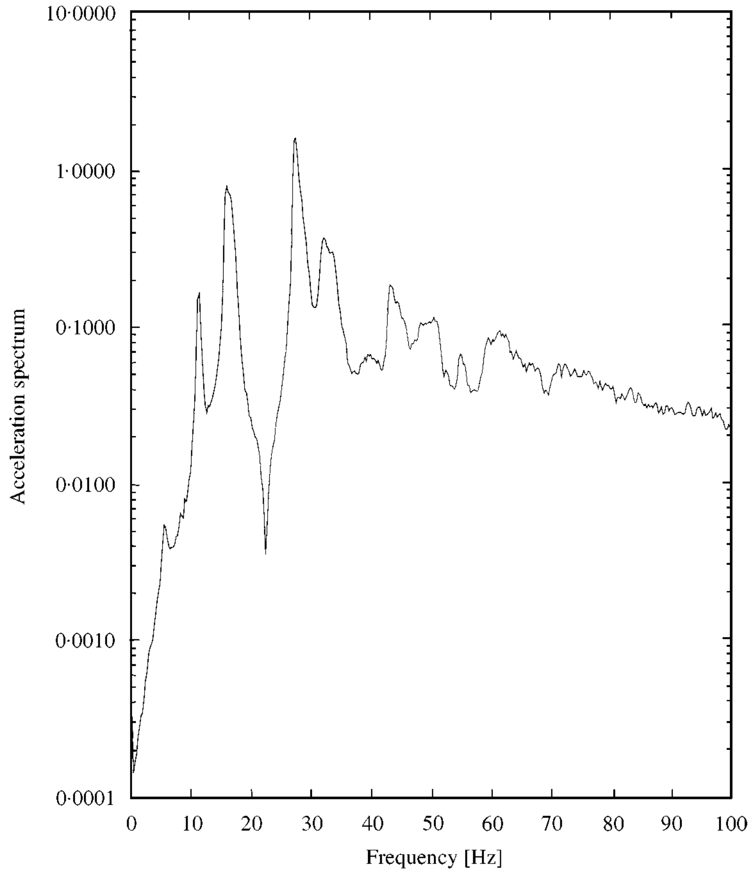

Figure 3. Spectrum from 2d.o.f. system with non-linear spring grounded.

6. CONCLUSIONS

The conclusions from these results are very simple. The poles for a non-linear system composite FRF appear to occur at well-defined combinations of the natural frequencies of the underlying non-linear system. As in the Sd.o.f. case, frequency shifts in the FRF peak at higher excitations can be explained in terms of the presence of the higher order poles. Because of the nature of the singularities as conjectured above, the implications for curve-fitting are not particularly hopeful unless the series solution can be truncated meaningfully at some finite order of P. The results above also shed further light on the experimental fact that the Hilbert transform test for non-linearity fails on FRFs obtained using random excitation.

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